October Math Gems

Problem of the week 1

§1 problems

Problem 1.1. In the following functional equation, solve for $f(x)$

$$f\left(x+\sqrt{x^2+1}\right)=\frac{x}{x+1}$$

Problem 1.2. If

$$\frac{\log(a)}{b-c}=\frac{\log(b)}{c-a}=\frac{\log(c)}{a-b}$$

Then compute $a^a b^b c^c$.

Problem 1.3. let r be a real number such that

$$\sqrt[3]{r}+\frac{1}{\sqrt[3]{r}}=3$$

Determine the value of

$$r^3+\frac{1}{r^3}$$

Problem 1.4. Let a and b be distinct real numbers. Solve the following equation

$$\sqrt{x-b^2}-\sqrt{x-a^2}=a-b$$

Problem 1.5. Solve for all real solutions of x

$$(x^2-7x+11)^{(x^2-13x+42)}=1$$

Problem 1.6. A club has 30 members. The positions of president, vice president, and treasurer will be assigned to 3 distinct members. What is the maximum number of distinct assignments that can be made?

Problem 1.7. If a, b, c are integers

$$\frac{ab}{a+b}=\frac{1}{3}, \qquad \frac{cb}{c+b}=\frac{1}{4}, \qquad \frac{ac}{a+c}=\frac{1}{5}$$

Find the value of

$$\frac{24abc}{ab+bc+ca}$$

Problem 1.8. The general solution of

$$\sin x - 3\sin^2 x + \sin^3 x = \cos x - 3\cos^2 x + \cos^3 x$$

Problem 1.9. If

$$f(n+3) = \frac{f(n)-1}{f(n)+1}, \qquad f(11) = 11$$

Find the value of $f(2003) =$

Problem 1.10. Find the least value of this algebraic expression

$$\sqrt{x^2+1} + \sqrt{(y-x)^2+4} + \sqrt{(z-y)^2+1} + \sqrt{(10-z)^2+9}$$

Problem 1.11. If $a+b+c=0$ then the value of

$$\frac{a^7+b^7+c^7}{abc(a^4+b^4+c^4)}$$

Problem 1.12.

$$\frac{1}{\log_2 m} + \frac{1}{\log_4 m} + \frac{1}{\log_8 m} \ldots\ldots + \frac{1}{\log_{2^n} m} = n(n+1)\alpha$$

Find α

Problem 1.13. In the triangle ABC, $AC = 2BC$, $\angle C = 90$ and D is the foot of the altitude from C onto AB. A circle diameter AD intersects the segment AC at E. Find $AE : EC$.

Problem 1.14. Evaluate the sum of the expression

$$\sum_{i=1}^{n} i(2)^i$$

Problem 1.15. If

$$\sin^{-1} x + \sin^{-1} y + \sin^{-1} z = \frac{3\pi}{2}$$

Then, find the value of

$$x^{100} + y^{100} + z^{100} - \frac{9}{x^{101} + y^{101} + z^{101}}$$

Problem 1.16. If four points are $A(6,3), B(-3,5), C(4,-2), P(x,y)$, then the ratio of areas between triangle ABC, PBC is

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Problem 1.17. Geeta and Babeeta play the following game. An inclusive integer between 0 and 999 is selected and given to Geeta. Whenever Geeta receives a number, he doubles it and passes the result to Babeeta. Whenever Babeeta receives a number, she adds 50 to it and passes the result to Geeta. The winner is the last person who produces a number less than 1000. Let N be the smallest initial number that results in a win for Geeta. What is the sum of the digits of N?

Problem 1.18. A function f is defined recursively by $f(1) = f(2) = 1$ and $f(n) = f(n-1) - f(n-2) + n$ for all integers $n \geq 3$ what is the sum of the digits of $f(2018)$?

Problem 1.19. Known that $a, b, c, d > 0$

$$\frac{1}{a+1} + \frac{2}{b+1} + \frac{3}{c+1} + \frac{4}{d+1} = x$$

$$\frac{a}{a+1} + \frac{2b}{b+1} + \frac{3c}{c+1} + \frac{4d}{d+1} = 4x$$

Then, $4x^2$ is equal to

Problem 1.20. If x is a complex number satisfying $x^2 + x + 1 = 0$, find the value of

$$x^{49} + x^{50} + x^{51} + x^{52} + x^{53}$$
